

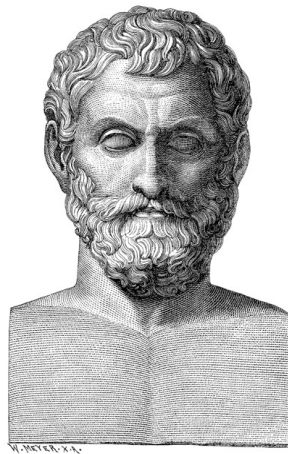
Addressing, with the concourse of Maple, diverse open issues dealing with the manifold and subtle concept of geometric locus

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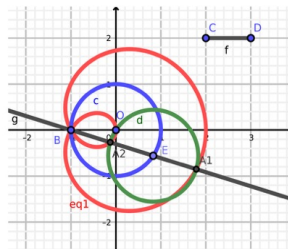
Granted by Comunidad de Madrid, Project IAxEM-CM, PHS-2024/PH-HUM-383



From Thales Theorem to octic curves:

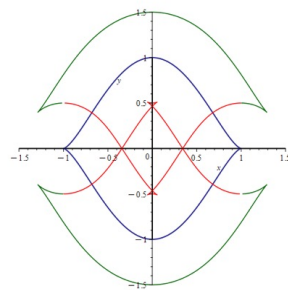
An illustrative journey, with the concurrence of Computer Algebra and Dynamic Geometry software, through the uncertain territory of locus computation

Tomas Recio, Thierry Dana-Picard
2024-10-31



Dynamic and automated constructions of plane curves

Thierry Dana-Picard, Zoltan Kovacs
2023-11-02



Envelopes of Circles Centered on a Kiss Curve

Thierry Dana-Picard, Daniel Tsirkin
2025-08-01

- Locus computation is a characteristic, defining feature of Dynamic Geometry (DG) software:

“There is a wide consensus among DG developers to **consider locus computation** as **one** of the five **basic properties** in the **DG** paradigm (together with dynamic transformation, measurement, free dragging and animation) (X. S. Gao, 1999)”*

- Already present, over sixty years ago, in the precursor of current DG programs, *SketchPad*

(*) Abanades, M.; Botana, F.; Montes, A.; Recio, T. (2014). An algebraic taxonomy for locus computation in dynamic geometry. *Computer-Aided Design* 56 , 22-33. DOI: 10.1016/j.cad.2014.06.008

X. S. Gao, Automated geometry diagram construction and engineering geometry, in: X. S. Gao, D. Wang, L. Yang (Eds.), *ADG 1998*, Vol. 1669700 of *Lecture Notes in Artificial Intelligence*, Springer, Heidelberg, 1999, pp. 232–257 3

Yet, current locus computation commands in DG programs are either

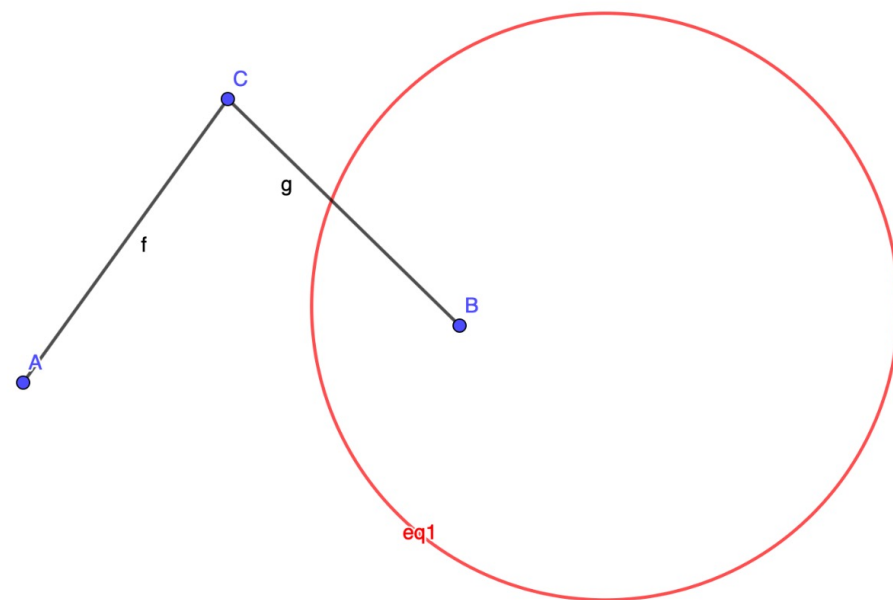
just visual plots (mover---tracer)

just approximate, numerical equations

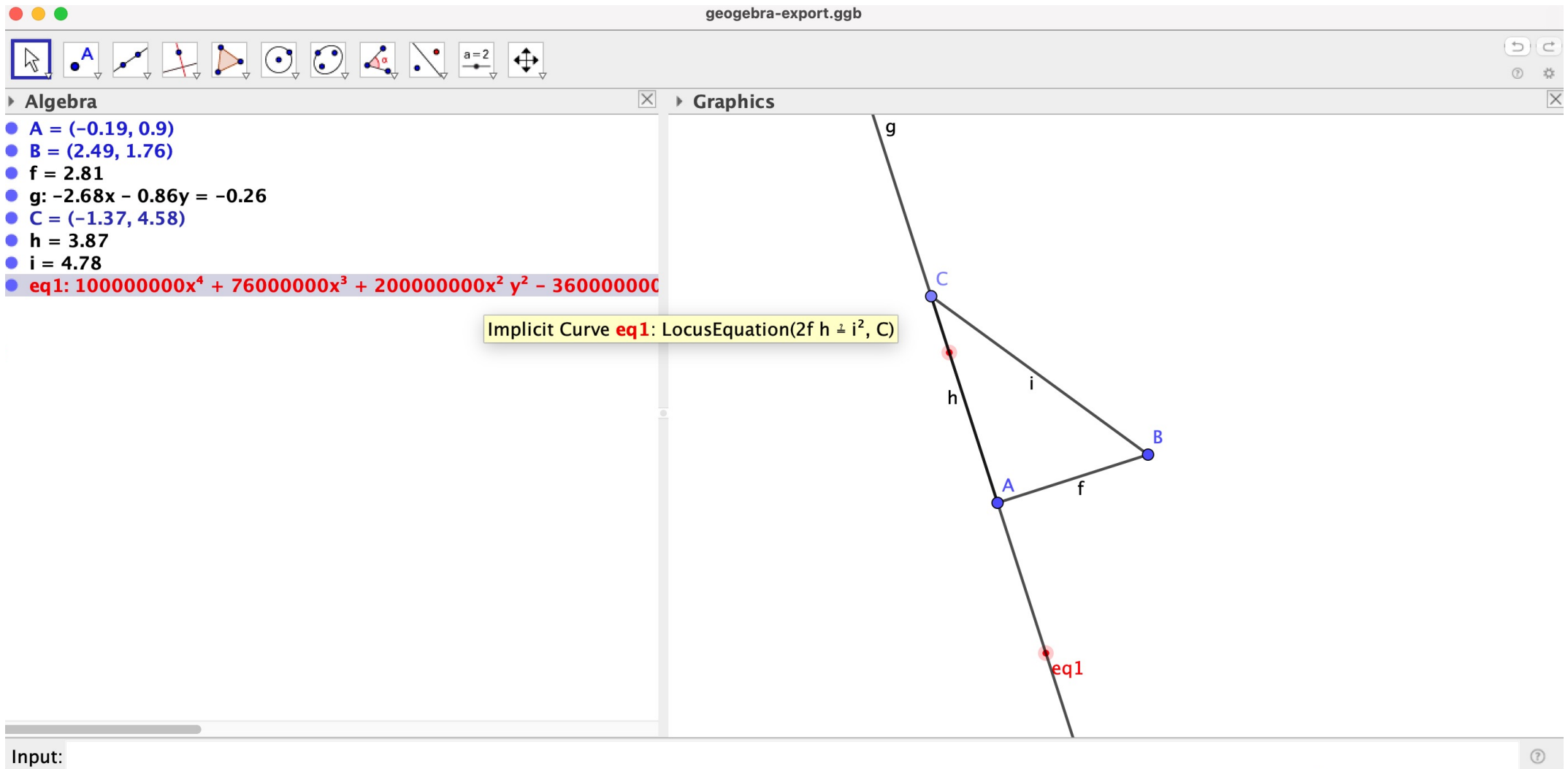
There is **NO symbolic approach to locus computation**, even in DG programs that include Computer Algebra features, such as GeoGebra



	A = (-9.35, -0.09)	
	B = (-2.61, 0.79)	
	C = (-6.19, 4.29)	
	f = Segment(A, C) → 5.4	
	g = Segment(C, B) → 5.01	
	eq1 : LocusEquation($f \stackrel{?}{=} 2 g, C$) → $15000x^2 + 10900x + 15000y^2 - 32500y = 288429$	
	Input...	







FINAL GOAL:

the future consideration and implementation of a
symbolic algorithmic approach to locus computation in
GeoGebra Discovery that would allow a sound, automated
verification of the correction and of the interpretation of its
output

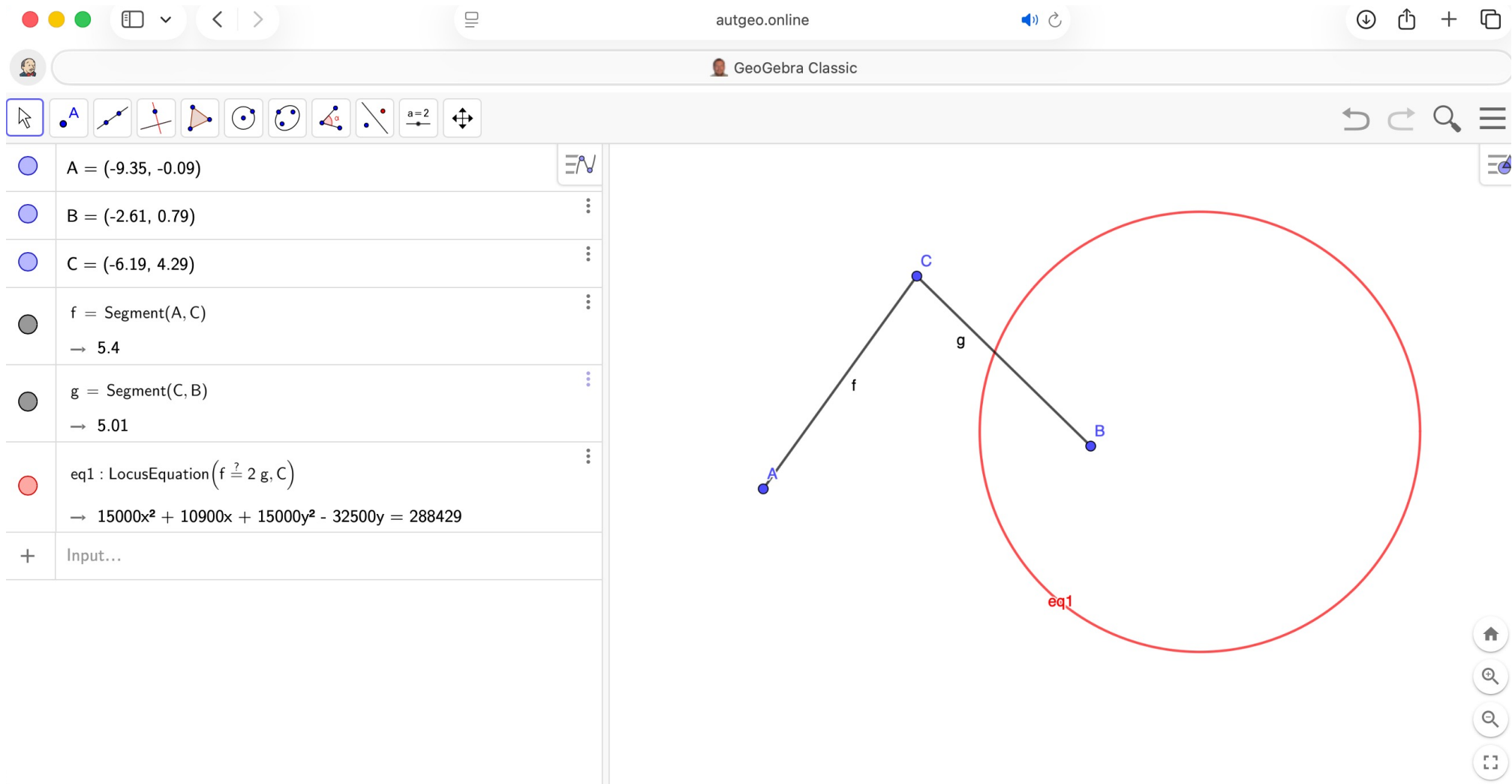
OUR ONGOING WORK:

experimenting with Maple the different algorithmic
possibilities to implement a symbolic locus equation
command

INVOLVED SYMBOLIC COMPUTATION ISSUES:

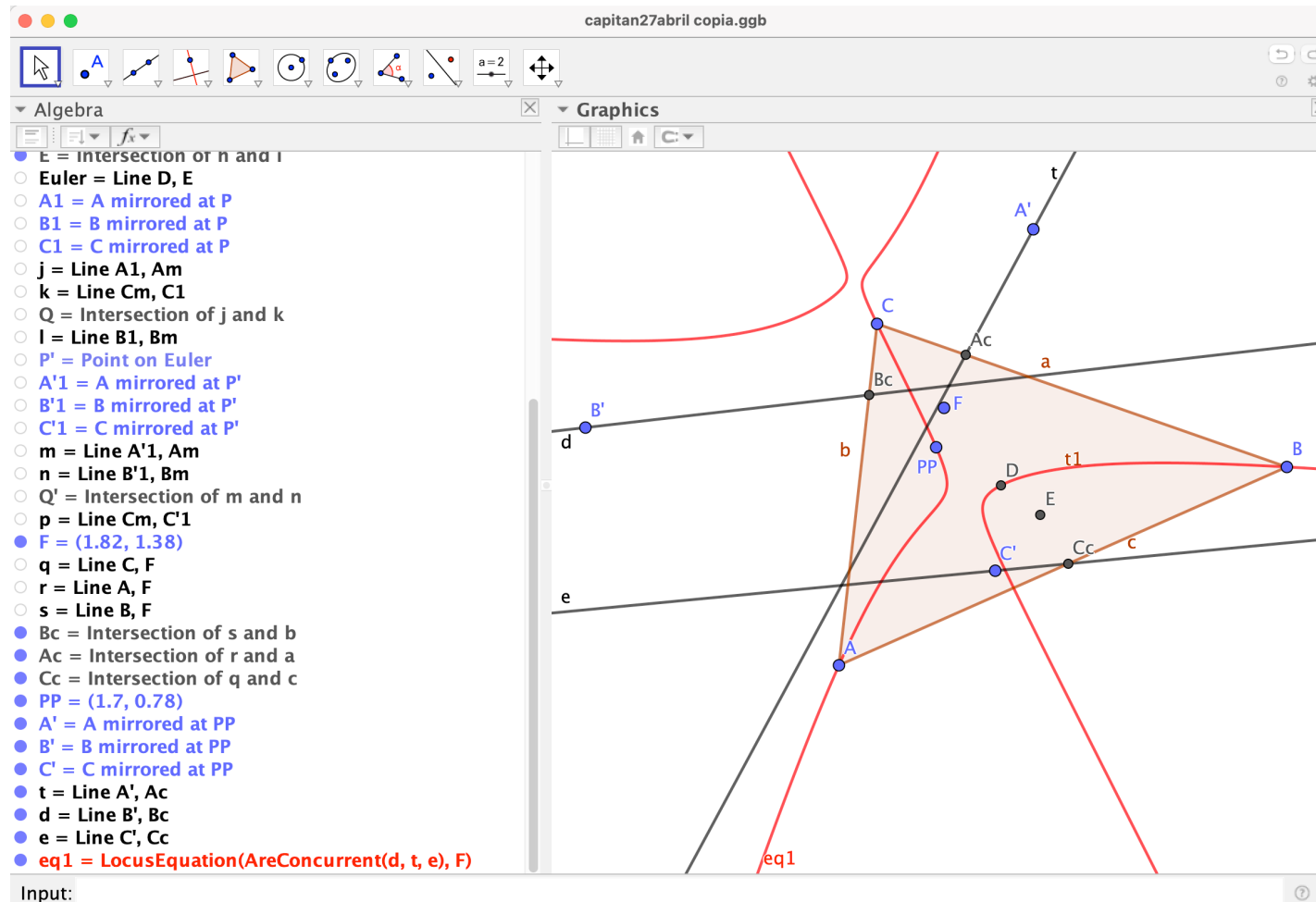
- How to deal with symbolic locus, i.e. with locus defined in the context of geometric constructions with arbitrary (parametric?) coordinates, and how to deal with the corresponding specialization for numerical values of the coordinates of the basic points of the same construction?
- Should we consider, in the EliminationIdeal protocol that is usually associated to locus computation, the parameters as variables or as elements of the field of coefficients?
- Should we eliminate over the coordinates of the locus point, or over a set of independent variables of the ideal describing the geometric construction where the locus point stands in?
- In what sense will we “discover” geometric statements by looking for the locus of some point in a construction that verifies a certain constraint?

- **Lemma 2.3.** Given an ideal $\alpha \subset K[p,x,y]$, for some sets of variables $\{p,x,y\}$, let $\mathfrak{b}$ be the elimination ideal $\alpha K[p,x,y] \cap K[p,y]$. Let $\mathfrak{c}$ be the extended ideal $\alpha K(p)[x,y]$ and let $\mathfrak{d}$ be the elimination ideal $\mathfrak{c} \cap K(p)[y]$. Then, the extension of $\mathfrak{b}$ coincides with $\mathfrak{d}$, i.e. $\mathfrak{b}^e = \mathfrak{b}K(p)[y] = \mathfrak{d}$.
- **Lemma 2.4.** Under the assumption that none of the primary components of α contain a non-zero polynomial in $K[p]$, we claim that the generators of the contraction $\mathfrak{d}^c$ of the elimination ideal $\mathfrak{d} = \alpha K(p)[x,y] \cap K(p)[y]$ to $K[p,y]$ generate as well $\mathfrak{b}$, the elimination ideal $\alpha K[p,x,y] \cap K[p,y]$.



Given a triangle ABC, we consider a general point F and the feet of the lines $r=AF$, $s=BF$, $q=CF$, (Ac , Bc , Cc , respectively). Next, we consider another arbitrary point PP and we consider $A'=A$ mirrored at PP, B' and C' (likewise). Then we define lines $t=A'Ac$, $d=B'Bc$, $e=C'Cc$.

And we wonder where to place F for having the concurrency of the three lines.



See <http://garciacapitan.epizy.com> and the generalization of a problema by D. Nguyen, from April 2025
<https://garciacapitan.blogspot.com/2025/04/generalizing-problem-nguyen085.html>

```
> Prepre:=<f1*v - f2*u, c1*v - c2*u + c2 - v, f1*q - f2*p + f2 - q
, c1*q - c2*p, c1*f2 - c1*s - c2*f1 + c2*r + f1*s - f2*r, s, aa1-2*
pp1, aa2-2*pp2, bb1-(2*pp1-1), bb2-2*pp2, cc1-(2*pp1-c1), cc2-(2*pp2
-c2), c2*t-1>;
```

$$\text{Prepre} := \langle s, aa1 - 2 pp1, aa2 - 2 pp2, bb2 - 2 pp2, bb1 - 2 pp1 + 1, cc1 - 2 pp1 + c1, \quad (5.4) \\ cc2 - 2 pp2 + c2, c2 t - 1, c1 q - c2 p, f1 v - f2 u, c1 v - c2 u + c2 - v, f1 q - f2 p \\ + f2 - q, c1 f2 - c1 s - c2 f1 + c2 r + f1 s - f2 r \rangle$$

```
> HilbertDimension(Prepre);
```

$$6 \quad (5.5)$$

```
> EliminationIdeal(Prepre, {c1, c2, f1, f2, pp1, pp2});
```

$$\langle 0 \rangle \quad (5.6)$$

```
> Condcond:=<f1*v - f2*u, c1*v - c2*u + c2 - v, f1*q - f2*p + f2 -
q , c1*q - c2*p, c1*f2 - c1*s - c2*f1 + c2*r + f1*s - f2*r, s, aa1
-2*pp1, aa2-2*pp2, bb1-(2*pp1-1), bb2-2*pp2, cc1-(2*pp1-c1), cc2-(2*
pp2-c2), aa1*v - aa1*y - aa2*u + aa2*x + u*y - v*x, bb1*q - bb1*y
- bb2*p + bb2*x + p*y - q*x, cc1*s - cc1*y - cc2*r + cc2*x + r*y -
s*x, c2*t-1>;
```

```
> HilbertDimension(Condcond);
5 (5.10)
```

```
> EliminationIdeal(Condcond, {c1, c2, f1, f2, pp1});
<0> (5.11)
```

```
> EliminationIdeal(Condcond, {c1, c2, f1, f2, pp1, pp2});
<-2 pp2 f2^3 c1^3 + 2 pp2^2 f2^2 c1^3 + 4 pp2 f1 f2^2 c1^2 c2 + 2 f2^3 pp1 c1^2 c2
- 6 pp2 f2^2 pp1 c1^2 c2 - 6 pp2^2 f1 f2^2 c1^2 + 6 pp2 f2^3 pp1 c1^2 - 2 pp2 f1^2 f2 c1 c2^2
- 2 pp2^2 f1^2 c1 c2^2 - 4 f1 f2^2 pp1 c1 c2^2 + 4 pp2 f1 f2 pp1 c1 c2^2 + 4 f2^2 pp1^2 c1 c2^2
+ 8 pp2^2 f1^2 f2 c1 c2 - 4 pp2 f1 f2^2 pp1 c1 c2 - 4 f2^3 pp1^2 c1 c2 + 2 f1^2 f2 pp1 c2^3
+ 2 pp2 f1^2 pp1 c2^3 - 4 f1 f2 pp1^2 c2^3 - 2 pp2^2 f1^3 c2^2 - 2 pp2 f1^2 f2 pp1 c2^2
+ 4 f1 f2^2 pp1^2 c2^2 - f2^3 c1^2 c2 + pp2 f2^2 c1^2 c2 + 2 f1 f2^2 c1 c2^2 + 2 pp2^2 f1 c1 c2^2
- 2 f2^2 pp1 c1 c2^2 - 2 pp2 f2 pp1 c1 c2^2 - 2 pp2 f1 f2^2 c1 c2 - 8 pp2^2 f1 f2 c1 c2
+ 2 f2^3 pp1 c1 c2 + 8 pp2 f2^2 pp1 c1 c2 + 6 pp2^2 f1 f2^2 c1 - 6 pp2 f2^3 pp1 c1
- f1^2 f2 c2^3 - pp2 f1^2 c2^3 + 2 f1 f2 pp1 c2^3 - 2 pp2 f1 pp1 c2^3 + 2 f2 pp1^2 c2^3
+ 2 pp2 f1^2 f2 c2^2 + 4 pp2^2 f1^2 c2^2 - 2 f1 f2^2 pp1 c2^2 - 4 f2^2 pp1^2 c2^2
- 4 pp2^2 f1^2 f2 c2 + 2 pp2 f1 f2^2 pp1 c2 + 2 f2^3 pp1^2 c2 - f2^3 c1 c2 + pp2 f2^2 c1 c2
+ 2 pp2 f2^3 c1 - 2 pp2^2 f2^2 c1 + pp2 f1 c2^3 - f2 pp1 c2^3 + f1 f2^2 c2^2 - 2 pp2 f1 f2 c2^2
- 2 pp2^2 f1 c2^2 + f2^2 pp1 c2^2 + 2 pp2 f2 pp1 c2^2 - 2 pp2 f1 f2^2 c2 + 4 pp2^2 f1 f2 c2
- 2 pp2 f2^2 pp1 c2>
```

```
> simplify(subs(f1=0,f2=0, -2*c1^3*f2^3*pp2 + 2*c1^3*f2^2*pp2^2 +
4*c1^2*c2*f1*f2^2*pp2 + 2*c1^2*c2*f2^3*pp1 - 6*c1^2*c2*f2^2*pp1*
pp2 - 6*c1^2*f1*f2^2*pp2^2 + 6*c1^2*f2^3*pp1*pp2 - 2*c1*c2^2*
f1^2*f2*pp2 - 2*c1*c2^2*f1^2*pp2^2 - 4*c1*c2^2*f1*f2^2*pp1 + 4*
c1*c2^2*f1*f2*pp1*pp2 + 4*c1*c2^2*f2^2*pp1^2 + 8*c1*c2*f1^2*f2*
pp2^2 - 4*c1*c2*f1*f2^2*pp1*pp2 - 4*c1*c2*f2^3*pp1^2 + 2*c2^3*
f1^2*f2*pp1 + 2*c2^3*f1^2*pp1*pp2 - 4*c2^3*f1*f2*pp1^2 - 2*c2^2*
f1^3*pp2^2 - 2*c2^2*f1^2*f2*pp1*pp2 + 4*c2^2*f1*f2^2*pp1^2 -
c1^2*c2*f2^3 + c1^2*c2*f2^2*pp2 + 2*c1*c2^2*f1*f2^2 + 2*c1*c2^2*
f1*pp2^2 - 2*c1*c2^2*f2^2*pp1 - 2*c1*c2^2*f2*pp1*pp2 - 2*c1*c2*
f1*f2^2*pp2 - 8*c1*c2*f1*f2*pp2^2 + 2*c1*c2*f2^3*pp1 + 8*c1*c2*
f2^2*pp1*pp2 + 6*c1*f1*f2^2*pp2^2 - 6*c1*f2^3*pp1*pp2 - c2^3*
f1^2*f2 - c2^3*f1^2*pp2 + 2*c2^3*f1*f2*pp1 - 2*c2^3*f1*pp1*pp2 +
2*c2^3*f2*pp1^2 + 2*c2^2*f1^2*f2*pp2 + 4*c2^2*f1^2*pp2^2 - 2*
c2^2*f1*f2^2*pp1 - 4*c2^2*f2^2*pp1^2 - 4*c2*f1^2*f2*pp2^2 + 2*c2*
f1*f2^2*pp1*pp2 + 2*c2*f2^3*pp1^2 - c1*c2*f2^3 + c1*c2*f2^2*pp2 +
2*c1*f2^3*pp2 - 2*c1*f2^2*pp2^2 + c2^3*f1*pp2 - c2^3*f2*pp1 +
c2^2*f1*f2^2 - 2*c2^2*f1*f2*pp2 - 2*c2^2*f1*pp2^2 + c2^2*f2^2*pp1
+ 2*c2^2*f2*pp1*pp2 - 2*c2*f1*f2^2*pp2 + 4*c2*f1*f2*pp2^2 - 2*c2*
f2^2*pp1*pp2));
```

0

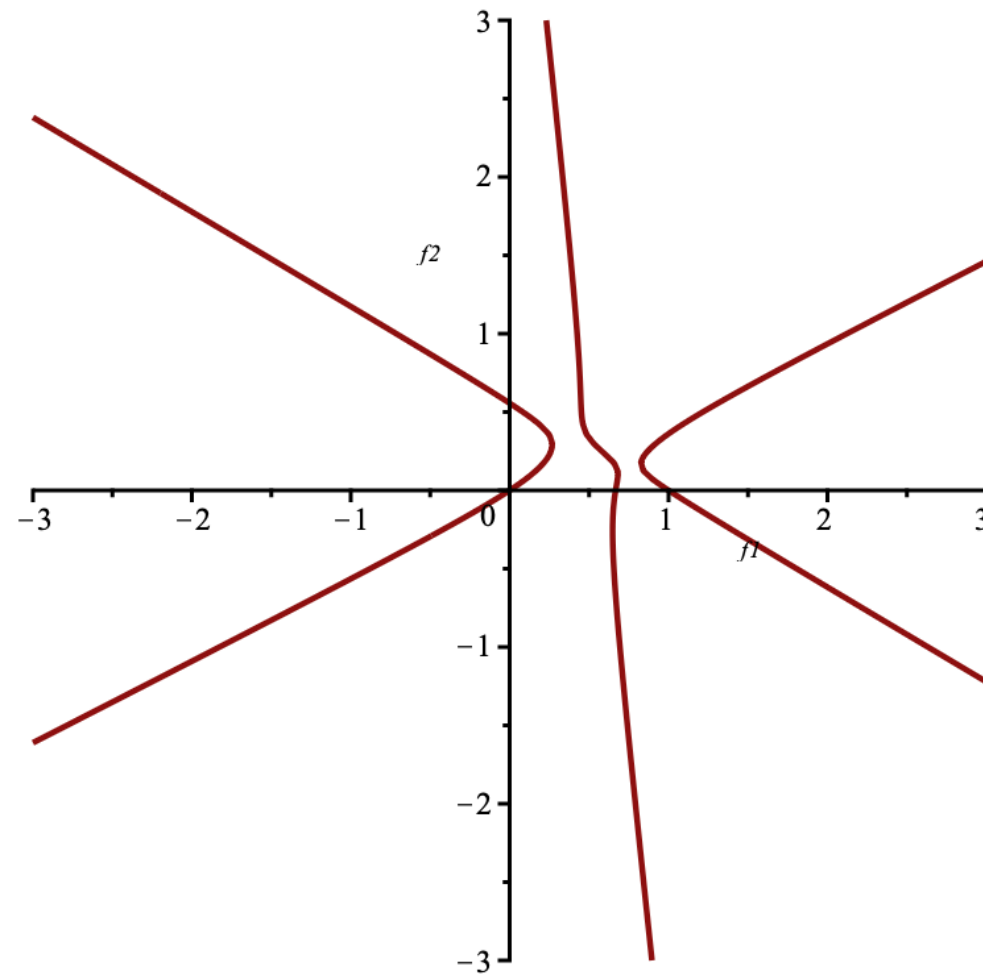
(5.14)

```
> simplify(subs(f1=1,f2=0, -2*c1^3*f2^3*pp2 + 2*c1^3*f2^2*pp2^2 +
4*c1^2*c2*f1*f2^2*pp2 + 2*c1^2*c2*f2^3*pp1 - 6*c1^2*c2*f2^2*pp1*
pp2 - 6*c1^2*f1*f2^2*pp2^2 + 6*c1^2*f2^3*pp1*pp2 - 2*c1*c2^2*
f1^2*f2*pp2 - 2*c1*c2^2*f1^2*pp2^2 - 4*c1*c2^2*f1*f2^2*pp1 + 4*
c1*c2^2*f1*f2*pp1*pp2 + 4*c1*c2^2*f2^2*pp1^2 + 8*c1*c2*f1^2*f2*
pp2^2 - 4*c1*c2*f1*f2^2*pp1*pp2 - 4*c1*c2*f2^3*pp1^2 + 2*c2^3*
f1^2*f2*pp1 + 2*c2^3*f1^2*pp1*pp2 - 4*c2^3*f1*f2*pp1^2 - 2*c2^2*
f1^3*pp2^2 - 2*c2^2*f1^2*f2*pp1*pp2 + 4*c2^2*f1*f2^2*pp1^2 -
c1^2*c2*f2^3 + c1^2*c2*f2^2*pp2 + 2*c1*c2^2*f1*f2^2 + 2*c1*c2^2*
f1*pp2^2 - 2*c1*c2^2*f2^2*pp1 - 2*c1*c2^2*f2*pp1*pp2 - 2*c1*c2*
f1*f2^2*pp2 - 8*c1*c2*f1*f2*pp2^2 + 2*c1*c2*f2^3*pp1 + 8*c1*c2*
f2^2*pp1*pp2 + 6*c1*f1*f2^2*pp2^2 - 6*c1*f2^3*pp1*pp2 - c2^3*
f1^2*f2 - c2^3*f1^2*pp2 + 2*c2^3*f1*f2*pp1 - 2*c2^3*f1*pp1*pp2 +
2*c2^3*f2*pp1^2 + 2*c2^2*f1^2*f2*pp2 + 4*c2^2*f1^2*pp2^2 - 2*
c2^2*f1*f2^2*pp1 - 4*c2^2*f2^2*pp1^2 - 4*c2*f1^2*f2*pp2^2 + 2*c2*
f1*f2^2*pp1*pp2 + 2*c2*f2^3*pp1^2 - c1*c2*f2^3 + c1*c2*f2^2*pp2 +
2*c1*f2^3*pp2 - 2*c1*f2^2*pp2^2 + c2^3*f1*pp2 - c2^3*f2*pp1 +
c2^2*f1*f2^2 - 2*c2^2*f1*f2*pp2 - 2*c2^2*f1*pp2^2 + c2^2*f2^2*pp1
+ 2*c2^2*f2*pp1*pp2 - 2*c2*f1*f2^2*pp2 + 4*c2*f1*f2*pp2^2 - 2*c2*
f2^2*pp1*pp2));
```

0

(5.15)

```
> with(plots, implicitplot):implicitplot(216*f1^3 - 36*f1^2*f2 -  
702*f1*f2^2 - 75*f2^3 - 360*f1^2 + 360*f1*f2 + 430*f2^2 + 144*f1  
- 216*f2,f1=-3..3,f2=-3..3);
```



```

> hh:=<f1*v - f2*u, c1*v - c2*u + c2 - v, f1*q - f2*p + f2 - q ,
c1*q - c2*p, c1*f2 - c1*s - c2*f1 + c2*r + f1*s - f2*r, s, aa1-2*
pp1, aa2-2*pp2, bb1-(2*pp1-1), bb2-2*pp2, cc1-(2*pp1-c1), cc2-(2*pp2
-c2), aa1*v - aa1*y - aa2*u + aa2*x + u*y - v*x, bb1*q - bb1*y -
bb2*p + bb2*x + p*y - q*x, c2*t-1, -2*c1^3*f2^3*pp2 + 2*c1^3*f2^2*
pp2^2 + 4*c1^2*c2*f1*f2^2*pp2 + 2*c1^2*c2*f2^3*pp1 - 6*c1^2*c2*
f2^2*pp1*pp2 - 6*c1^2*f1*f2^2*pp2^2 + 6*c1^2*f2^3*pp1*pp2 - 2*c1*
c2^2*f1^2*f2*pp2 - 2*c1*c2^2*f1^2*pp2^2 - 4*c1*c2^2*f1*f2^2*pp1 +
4*c1*c2^2*f1*f2*pp1*pp2 + 4*c1*c2^2*f2^2*pp1^2 + 8*c1*c2*f1^2*f2*
pp2^2 - 4*c1*c2*f1*f2^2*pp1*pp2 - 4*c1*c2*f2^3*pp1^2 + 2*c2^3*
f1^2*f2*pp1 + 2*c2^3*f1^2*pp1*pp2 - 4*c2^3*f1*f2*pp1^2 - 2*c2^2*
f1^3*pp2^2 - 2*c2^2*f1^2*f2*pp1*pp2 + 4*c2^2*f1*f2^2*pp1^2 -
c1^2*c2*f2^3 + c1^2*c2*f2^2*pp2 + 2*c1*c2^2*f1*f2^2 + 2*c1*c2^2*
f1*pp2^2 - 2*c1*c2^2*f2^2*pp1 - 2*c1*c2^2*f2*pp1*pp2 - 2*c1*c2*
f1*f2^2*pp2 - 8*c1*c2*f1*f2*pp2^2 + 2*c1*c2*f2^3*pp1 + 8*c1*c2*
f2^2*pp1*pp2 + 6*c1*f1*f2^2*pp2^2 - 6*c1*f2^3*pp1*pp2 - c2^3*
f1^2*f2 - c2^3*f1^2*pp2 + 2*c2^3*f1*f2*pp1 - 2*c2^3*f1*pp1*pp2 +
2*c2^3*f2*pp1^2 + 2*c2^2*f1^2*f2*pp2 + 4*c2^2*f1^2*pp2^2 - 2*
c2^2*f1*f2^2*pp1 - 4*c2^2*f2^2*pp1^2 - 4*c2*f1^2*f2*pp2^2 + 2*c2*
f1*f2^2*pp1*pp2 + 2*c2*f2^3*pp1^2 - c1*c2*f2^3 + c1*c2*f2^2*pp2 +
2*c1*f2^3*pp2 - 2*c1*f2^2*pp2^2 + c2^3*f1*pp2 - c2^3*f2*pp1 +
c2^2*f1*f2^2 - 2*c2^2*f1*f2*pp2 - 2*c2^2*f1*pp2^2 + c2^2*f2^2*pp1
+ 2*c2^2*f2*pp1*pp2 - 2*c2*f1*f2^2*pp2 + 4*c2*f1*f2*pp2^2 - 2*c2*
f2^2*pp1*pp2>;

```

$hh := \langle s, aa1 - 2 pp1, aa2 - 2 pp2, bb2 - 2 pp2, bb1 - 2 pp1 + 1, cc1 - 2 pp1 + c1, cc2$ (6.4)

$$\begin{aligned}
& - 2 pp2 + c2, c2 t - 1, c1 q - c2 p, f1 v - f2 u, c1 v - c2 u + c2 - v, f1 q - f2 p + f2 \\
& - q, aa1 v - aa1 y - aa2 u + aa2 x + u y - v x, bb1 q - bb1 y - bb2 p + bb2 x + p y \\
& - q x, c1 f2 - c1 s - c2 f1 + c2 r + f1 s - f2 r, -2 c1^3 f2^3 pp2 + 2 c1^3 f2^2 pp2^2 \\
& + 4 c1^2 c2 f1 f2^2 pp2 + 2 c1^2 c2 f2^3 pp1 - 6 c1^2 c2 f2^2 pp1 pp2 - 6 c1^2 f1 f2^2 pp2^2 \\
& + 6 c1^2 f2^3 pp1 pp2 - 2 c1 c2^2 f1^2 f2 pp2 - 2 c1 c2^2 f1^2 pp2^2 - 4 c1 c2^2 f1 f2^2 pp1 \\
& + 4 c1 c2^2 f1 f2 pp1 pp2 + 4 c1 c2^2 f2^2 pp1^2 + 8 c1 c2 f1^2 f2 pp2^2 \\
& - 4 c1 c2 f1 f2^2 pp1 pp2 - 4 c1 c2 f2^3 pp1^2 + 2 c2^3 f1^2 f2 pp1 + 2 c2^3 f1^2 pp1 pp2 \\
& - 4 c2^3 f1 f2 pp1^2 - 2 c2^2 f1^3 pp2^2 - 2 c2^2 f1^2 f2 pp1 pp2 + 4 c2^2 f1 f2^2 pp1^2 \\
& - c1^2 c2 f2^3 + c1^2 c2 f2^2 pp2 + 2 c1 c2^2 f1 f2^2 + 2 c1 c2^2 f1 pp2^2 - 2 c1 c2^2 f2^2 pp1 \\
& - 2 c1 c2^2 f2 pp1 pp2 - 2 c1 c2 f1 f2^2 pp2 - 8 c1 c2 f1 f2 pp2^2 + 2 c1 c2 f2^3 pp1
\end{aligned}$$

- The ideal hh is of **dimension 5**, and **C, PP, f1** can be considered (this is human intuition) as the "**ruling**" **independent variables** (it is reasonable, since there is an equation (the condition) holding between C, F, PP).
- Now we **add the negation of the thesis**

$$(cc1*s - cc1*y - cc2*r + cc2*x + r*y - s*x)*g-1$$

with g dumb variable, and check (after 8 minutes computation) that **the elimination** of the new ideal ll over the C, PP, f1 variables **is not zero** .

$$\begin{aligned}
& > \text{factor}(-8*c1^4*f1^2*pp2^4 + 8*c1^3*c2*f1^3*pp2^3 + 16*c1^3*c2* \\
& \quad f1^2*pp1*pp2^3 - 16*c1^2*c2^2*f1^3*pp1*pp2^2 - 8*c1^2*c2^2*f1^2* \\
& \quad pp1^2*pp2^2 - 8*c1^2*c2*f1^4*pp2^3 + 8*c1^2*f1^4*pp2^4 + 8*c1* \\
& \quad c2^3*f1^3*pp1^2*pp2 + 16*c1*c2^2*f1^4*pp1*pp2^2 - 16*c1*c2*f1^4* \\
& \quad pp1*pp2^3 - 8*c2^3*f1^4*pp1^2*pp2 + 8*c2^2*f1^4*pp1^2*pp2^2 + 8* \\
& \quad c1^4*f1*pp2^4 - 20*c1^3*c2*f1^2*pp2^3 - 16*c1^3*c2*f1*pp1*pp2^3 + \\
& \quad 16*c1^3*f1^2*pp2^4 + 8*c1^2*c2^2*f1^3*pp2^2 + 32*c1^2*c2^2*f1^2* \\
& \quad pp1*pp2^2 + 8*c1^2*c2^2*f1*pp1^2*pp2^2 + 4*c1^2*c2*f1^3*pp2^3 - \\
& \quad 24*c1^2*c2*f1^2*pp1*pp2^3 - 16*c1^2*f1^3*pp2^4 - 8*c1*c2^3*f1^3* \\
& \quad pp1*pp2 - 12*c1*c2^3*f1^2*pp1^2*pp2 - 8*c1*c2^2*f1^4*pp2^2 - 16* \\
& \quad c1*c2^2*f1^3*pp1*pp2^2 + 8*c1*c2^2*f1^2*pp1^2*pp2^2 + 16*c1*c2* \\
& \quad f1^4*pp2^3 + 32*c1*c2*f1^3*pp1*pp2^3 - 8*c1*f1^4*pp2^4 + 8*c2^3* \\
& \quad f1^4*pp1*pp2 + 12*c2^3*f1^3*pp1^2*pp2 - 16*c2^2*f1^4*pp1*pp2^2 - \\
& \quad 16*c2^2*f1^3*pp1^2*pp2^2 + 8*c2*f1^4*pp1*pp2^3 + 12*c1^3*c2*f1* \\
& \quad pp2^3 - 16*c1^3*f1*pp2^4 - 14*c1^2*c2^2*f1^2*pp2^2 - 16*c1^2* \\
& \quad c2^2*f1*pp1*pp2^2 + 20*c1^2*c2*f1^2*pp2^3 + 24*c1^2*c2*f1*pp1* \\
& \quad pp2^3 + 2*c1*c2^3*f1^3*pp2 + 12*c1*c2^3*f1^2*pp1*pp2 + 4*c1*c2^3* \\
& \quad f1*pp1^2*pp2 + 8*c1*c2^2*f1^3*pp2^2 - 12*c1*c2^2*f1^2*pp1*pp2^2 - \\
& \quad 8*c1*c2^2*f1*pp1^2*pp2^2 - 28*c1*c2*f1^3*pp2^3 - 8*c1*c2*f1^2* \\
& \quad pp1*pp2^3 + 16*c1*f1^3*pp2^4 - 2*c2^3*f1^4*pp2 - 12*c2^3*f1^3* \\
& \quad pp1*pp2 - 4*c2^3*f1^2*pp1^2*pp2 + 6*c2^2*f1^4*pp2^2 + 28*c2^2* \\
& \quad f1^3*pp1*pp2^2 + 8*c2^2*f1^2*pp1^2*pp2^2 - 4*c2*f1^4*pp2^3 - 16* \\
& \quad c2*f1^3*pp1*pp2^3 + 6*c1^2*c2^2*f1*pp2^2 - 16*c1^2*c2*f1*pp2^3 + \\
& \quad 8*c1^2*f1*pp2^4 - 3*c1*c2^3*f1^2*pp2 - 4*c1*c2^3*f1*pp1*pp2 + 4* \\
& \quad c1*c2^2*f1^2*pp2^2 + 12*c1*c2^2*f1*pp1*pp2^2 + 8*c1*c2*f1^2*pp2^3 \\
& \quad - 8*c1*c2*f1*pp1*pp2^3 - 8*c1*f1^2*pp2^4 + 3*c2^3*f1^3*pp2 + 4* \\
& \quad c2^3*f1^2*pp1*pp2 - 10*c2^2*f1^3*pp2^2 - 12*c2^2*f1^2*pp1*pp2^2 + \\
& \quad 8*c2*f1^3*pp2^3 + 8*c2*f1^2*pp1*pp2^3 + c1*c2^3*f1*pp2 - 4*c1* \\
& \quad c2^2*f1*pp2^2 + 4*c1*c2*f1*pp2^3 - c2^3*f1^2*pp2 + 4*c2^2*f1^2* \\
& \quad pp2^2 - 4*c2*f1^2*pp2^3); \\
& -f1\ pp2\ (f1 - 1)\ (c1 - f1)\ (2\ c1\ pp2 - 2\ pp1\ c2 + c2 - 2\ pp2)\ (2\ c1\ pp2 - 2\ pp1\ c2 \\
& \quad + c2)\ (2\ c1\ pp2 - 2\ c2\ f1 + 2\ f1\ pp2 + c2 - 2\ pp2)
\end{aligned} \tag{6.9}$$

Thank you for your attention
Any questions or comments?