Effective Methods in Algebraic Geometry 2026

Durham, 7th July 2026

The ShowProof command in GeoGebra: Automated ranking and derivation of geometric statements

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*Supported by the **IAxEM Project "Inteligencia Aumentada en Educación Matemática"** CM/PHS-2024/PH-HUM-383. Comunidad de Madrid*

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GeoGebra and **GeoGebra Discovery**

GeoGebra Discovery (**off-line**): https://github.com/kovzol/geogebra/releases

GeoGebra Discovery (**on-line**). http://www.autgeo.online/

(or Google "autgeo.online")

Why GeoGebra?

- Dynamic Geometry Software (DGS) with easy use online, smartphones, tablets…
- 100+ million users worldwide, 80+ languages.
- Suitable for educational settings.

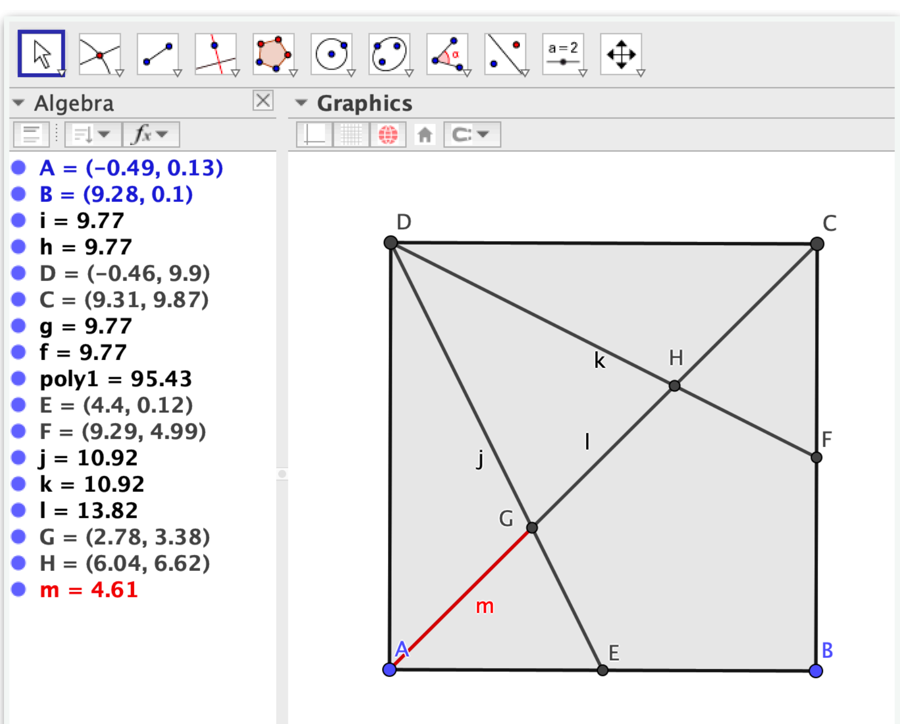
GeoGebra Discovery is an experimental version of GeoGebra extending existing methods and developing new Automated Reasoning Tools (ART).

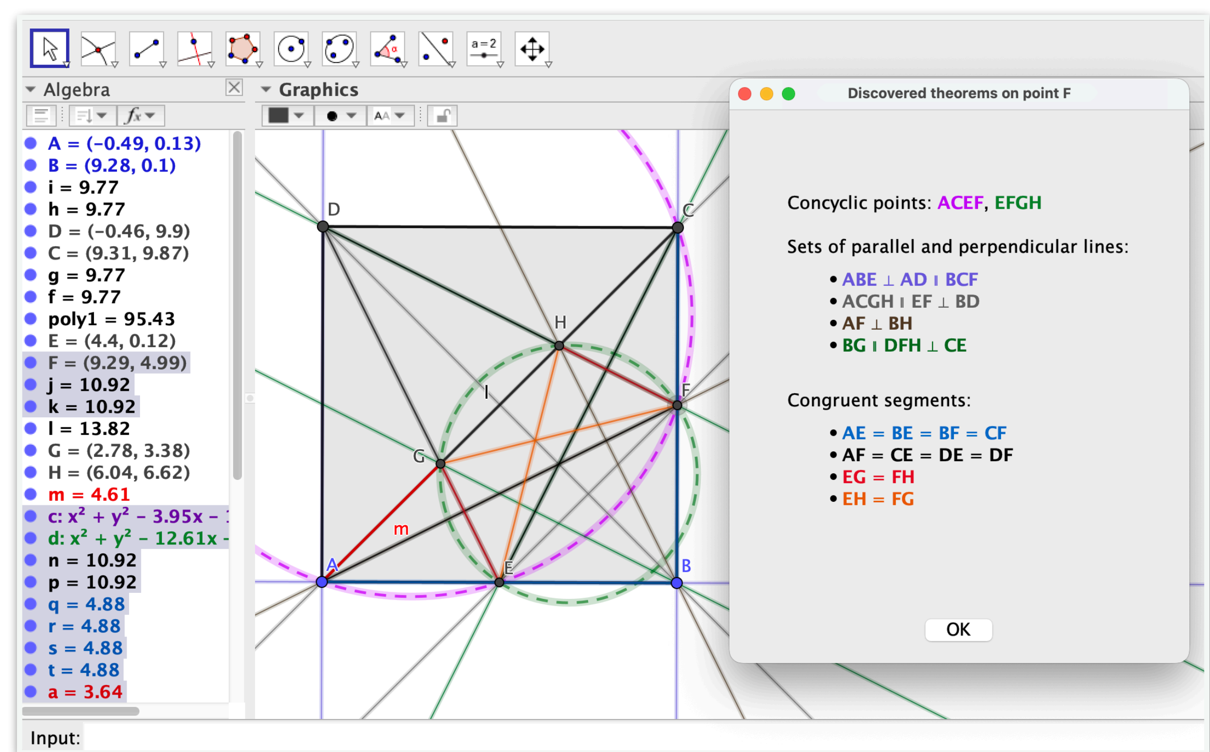
- Effective algorithms implemented (GIAC, Tarski), and exportable to other softwares (Maple, Mathematica…) for further analysis.

Automated Reasoning Tools in GeoGebra Discovery

<u>Given a construction</u> (made by the user) GeoGebra Discovery can:

- Find the **relation** between two given geometric objects (lengths, algebraic number ratios, or inequality relations)
- Finding **missing hypotheses** for a given relation to hold true.
- **Check the truth/failure** of a conjectured statement (by the user) and give a certified step-by-step algebraic proof.
- **Discover automatically** all possible relations of certain kind (co-circularity, parallelism, perpendicularity, etc) among the objects.





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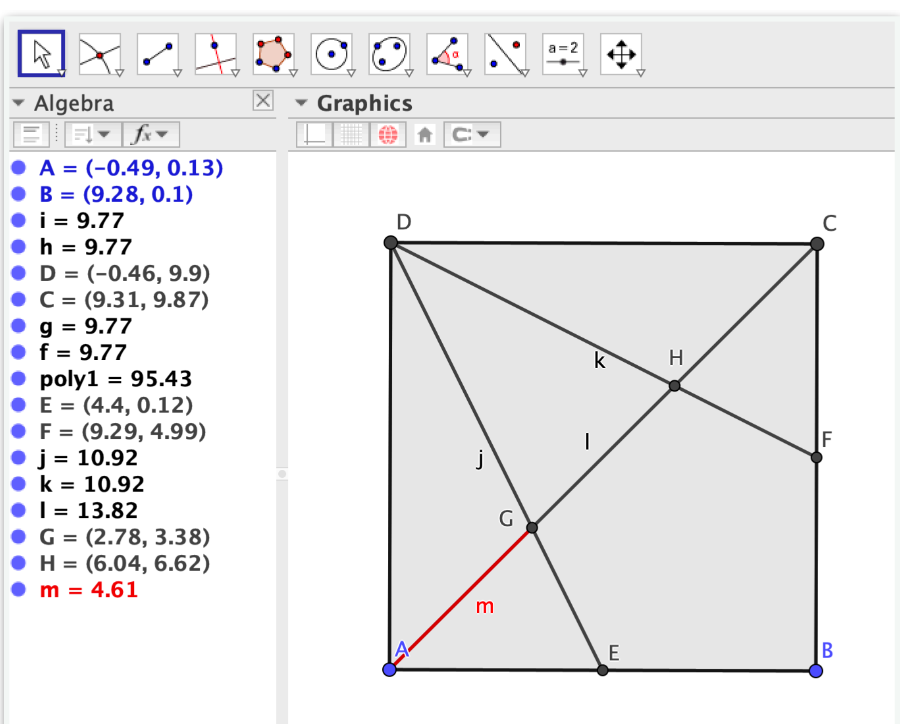
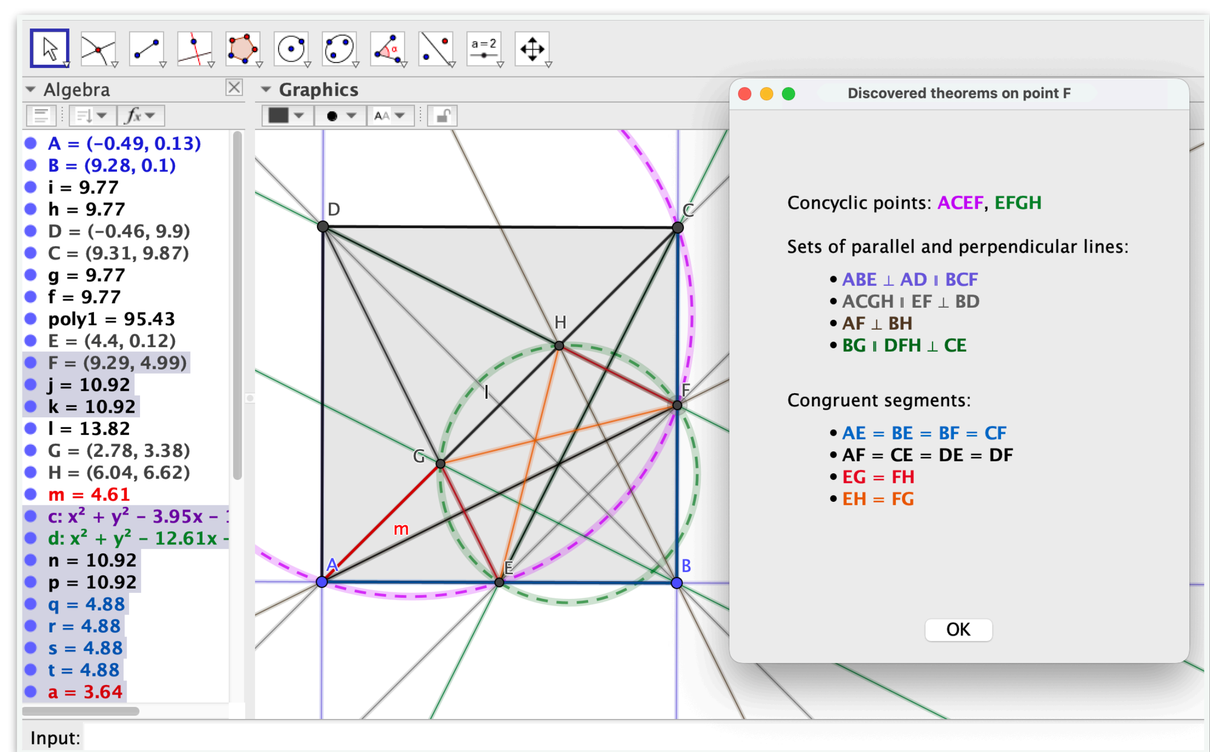
Today



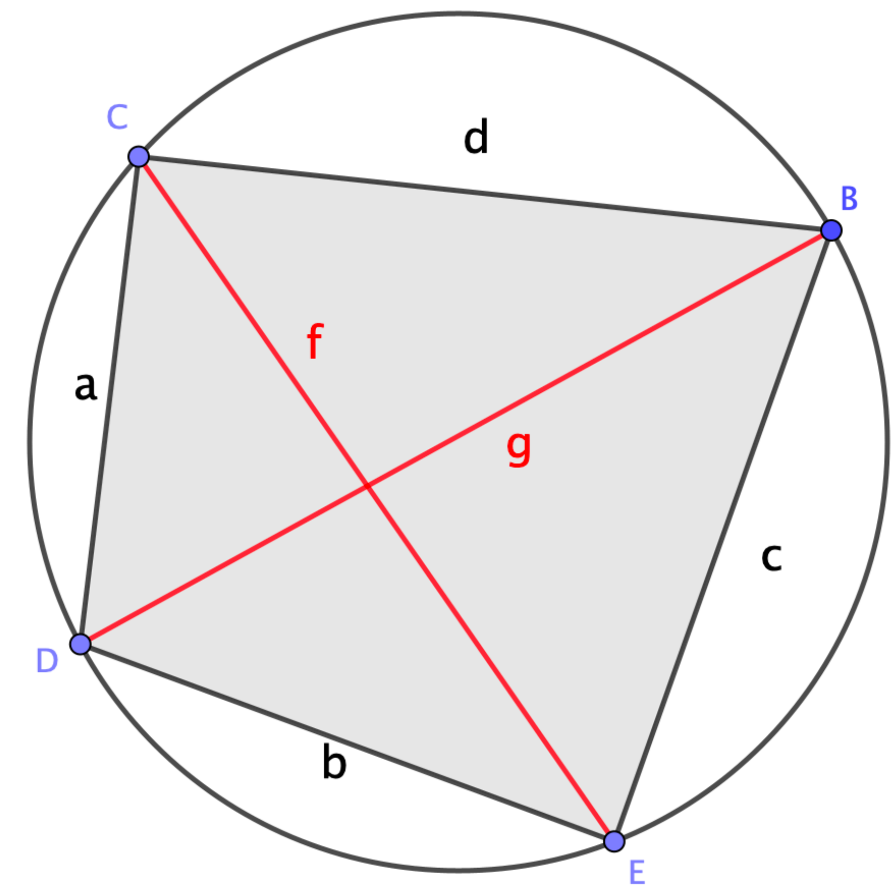


Example

Ptolemy's Theorem

A quadrilateral is cyclic $\iff$ $fg = ac + bd$

The product of diagonals equals the sum of products of opposite sides



Let us use some ART tools in GeoGebra Discovery with this example:

LocusEquation, Prove and ***ShowProof***.

The ShowProof Command

[Wu, Kapur,… 80s]

[Recio-Vélez ’99,…]

[Boutry-Braun-Narboux, 2019]

“Idea”

Prove the geometric statement: Hypotheses $H \implies$ Thesis T

Algebraically: $s_1 = \ldots = s_n = 0 \implies T = 0$

$s_1, \ldots, s_n, T \in \mathbb{Q}[v_1, \ldots, v_r], \quad v_i =$ point coords

Proof by contradiction: $I = < s_1, \ldots, s_n, s_{n+1} >$ where $s_{n+1} = zT - 1$

Thesis Negation

“Rabinowistch trick”

If Gröbner $G(I) = 1 \implies V(I) = \varnothing$

True statement

The ShowProof Command

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True statement

1. **Difficulty degree** of a statement:

$$1 \in I \Longrightarrow 1 = a_1 s_1 + \ldots + a_n s_n + a_{n+1} s_{n+1}$$

$$\boxed{T^k = a_1' s_1 + \ldots + a_n' s_n, \text{ for some } k}$$

Thesis in terms of the Hypotheses

Difficulty degree

$d = \text{maxdeg}(a_i)$

The ShowProof Command

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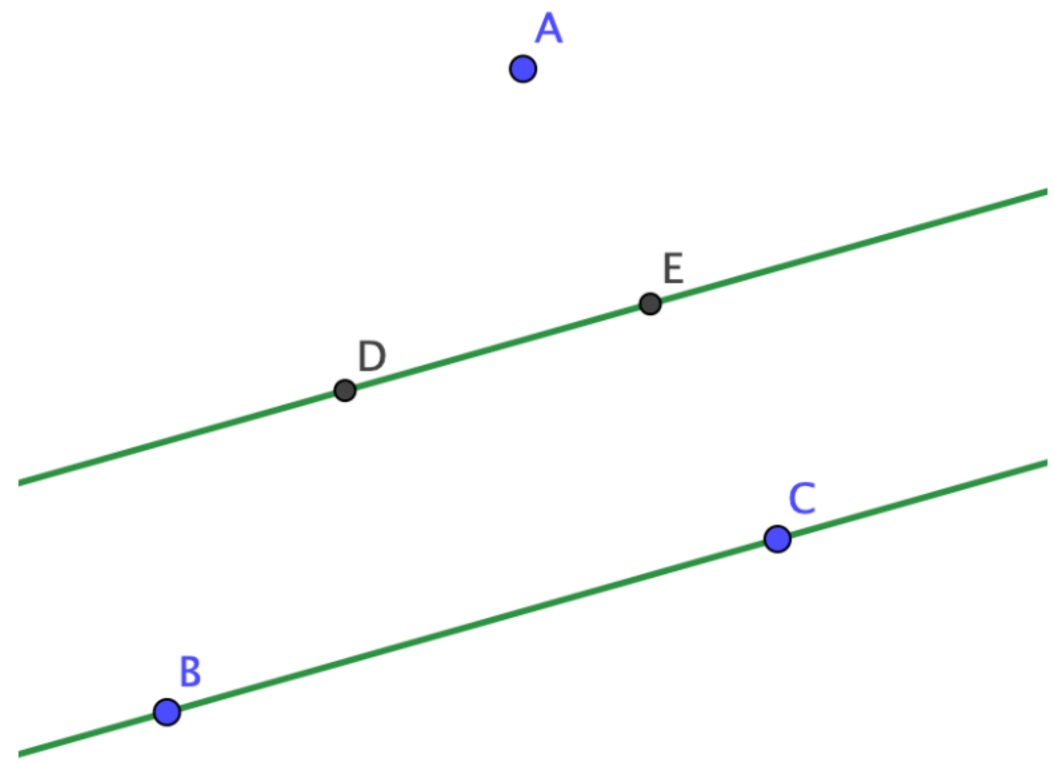
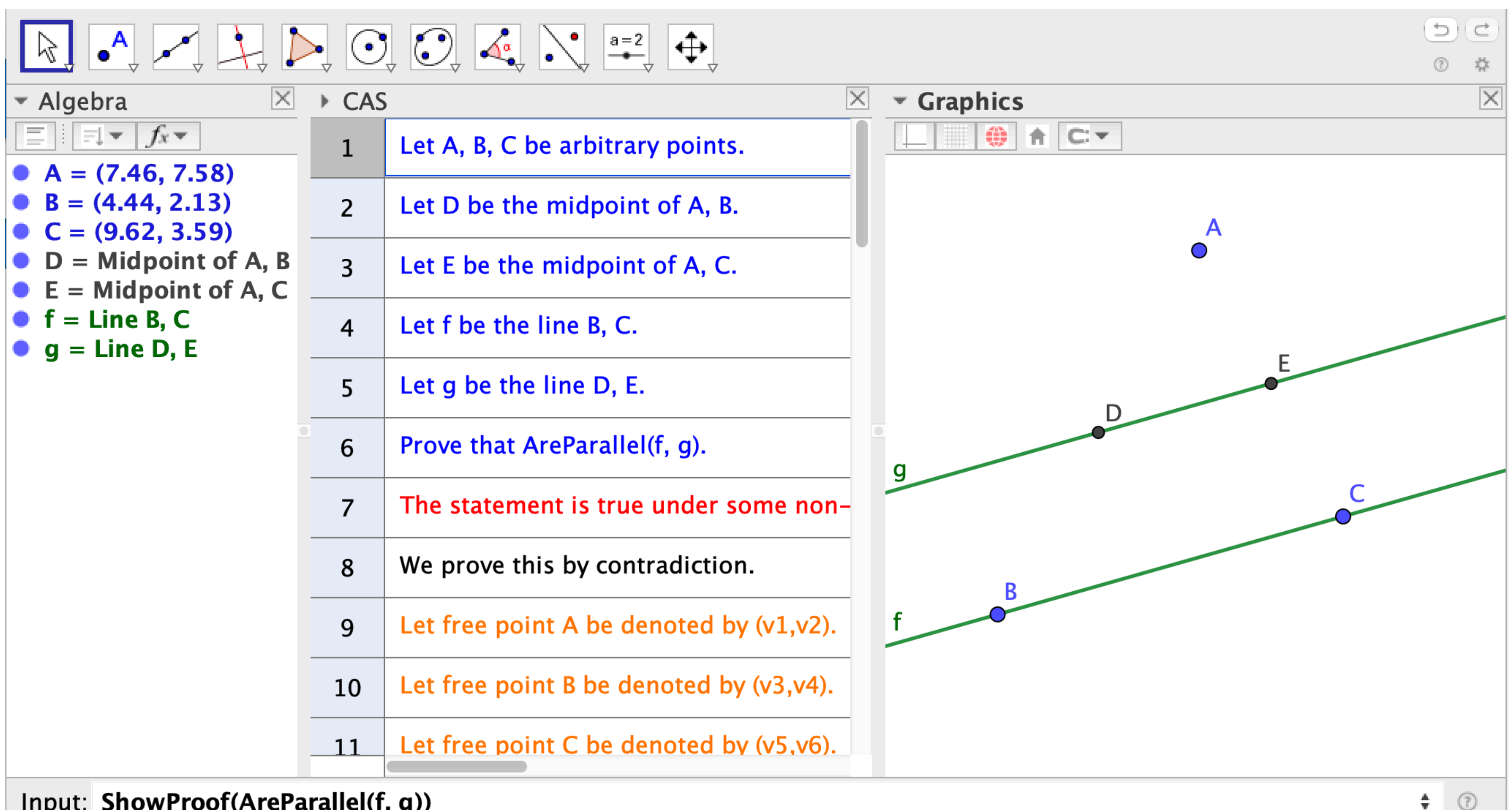
2. **Automated derivation of Theorems** inspired by [Peng-Zhang-Chen-Liu, 2025]

Example



Thales Midpoint Theorem

Given 3 points A, B and C.
The line joining the midpoints D, E of AB and AC is parallel to the line BC



Difficulty degree = 2

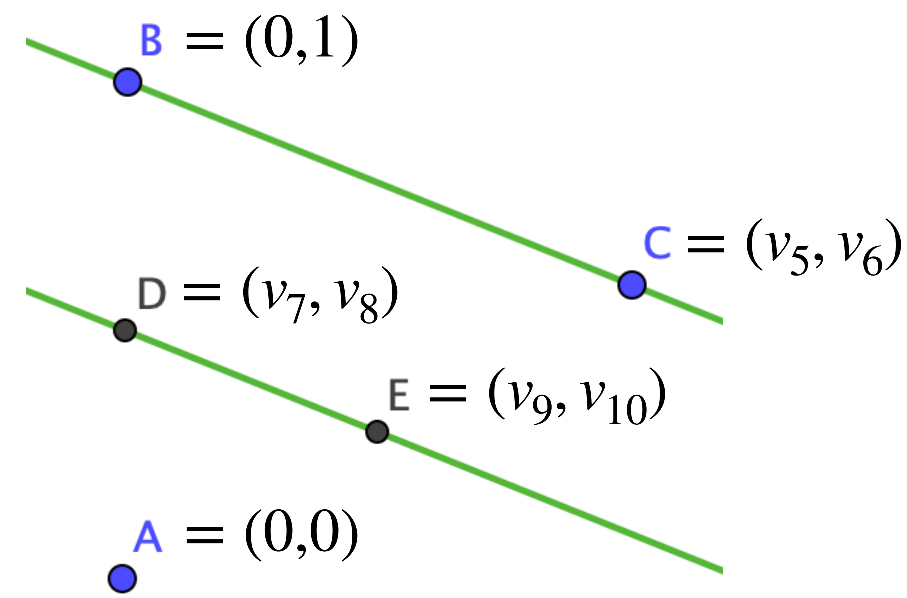
Hypotheses:

D is the midpoint of AB

$$s_1 = 2v_7; \quad s_2 = -1 + 2v_8$$

E is the midpoint of AC

$$s_3 = -v_5 + 2v_9; \quad s_4 = -v_6 + 2v_{10}$$



Thesis:

The line DE is parallel to the line BC

$$T = -v_4v_7 + v_6v_7 + v_3v_8 - v_5v_8 + v_4v_9 - v_6v_9 - v_3v_{10} + v_5v_{10}$$

Negation of the thesis: $s_5 = v_{11}T - 1$

$I = < s_1, s_2, s_3, s_4, s_5 >$. Gröbner $G(I) = 1$, the result is TRUE

$$1 \in I \implies 1 = s_1 \cdot (v_{10}v_{11} - v_{11}/2) + s_2 \cdot (-v_{11}v_9) + s_3 \cdot (-v_{10}v_{11} + v_{11}v_8) +$$
$$+ s_4 \cdot (-v_{11}v_7 + v_{11}v_9) + s_5 \cdot (-1)$$

Difficulty degree = 2

$$(v_{11} = \frac{1}{T})$$

$$\boxed{T = s_1 \cdot (-v_{10} + 1/2) + s_2 \cdot v_9 + s_3 \cdot (v_{10} - v_8) + s_4 \cdot (v_7 - v_9)}$$

Thesis in terms of hypotheses

Derivation 1

New Thesis

New Hypotheses

$$s_2 \cdot v_9 \; = \; T \; - \; s_1 \cdot (v_{10} \; + \; 1/2) \; - \; s_3 \cdot (v_{10} - v_8) \; - \; s_4 \cdot (v_7 - v_9)$$

Derivation 1

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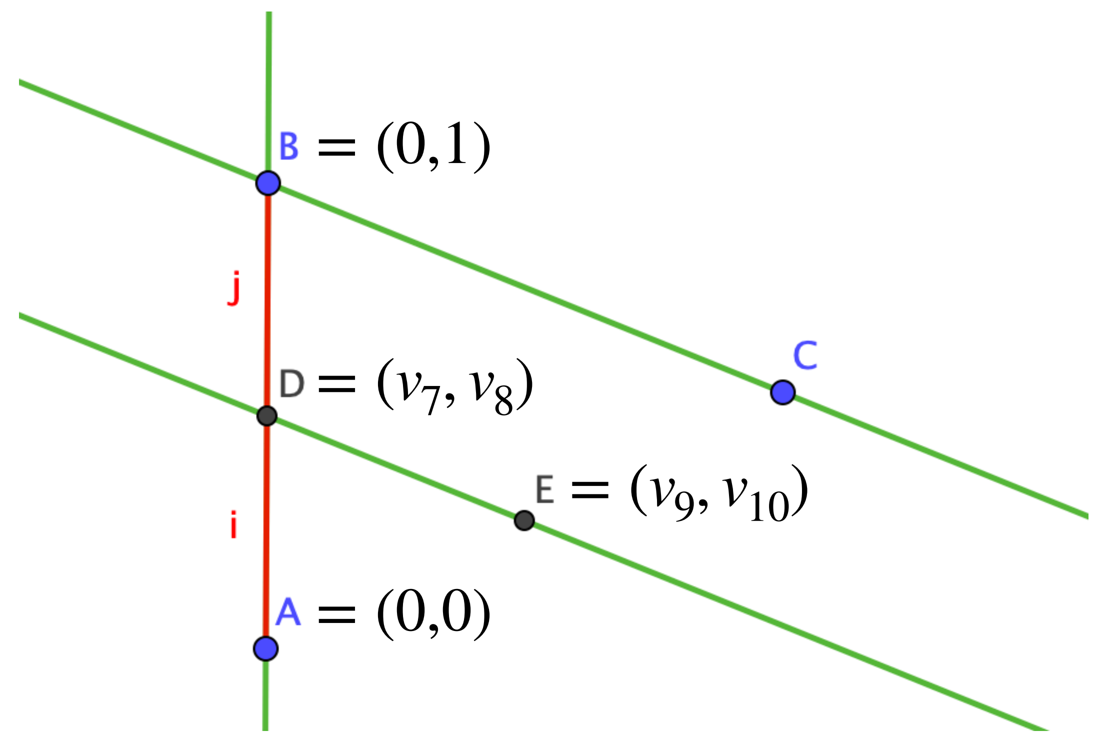
$$s_2 \cdot v_9 = T - s_1 \cdot (v_{10} + 1/2) - s_3 \cdot (v_{10} - v_8) - s_4 \cdot (v_7 - v_9)$$

$v_8 = 1/2$ $\Longleftrightarrow$ $D \in$ line AB

$DE \,//\, BC$

$v_7 = 0$ $\Longleftrightarrow$ $D \in$ line AB

E is the midpoint of AC



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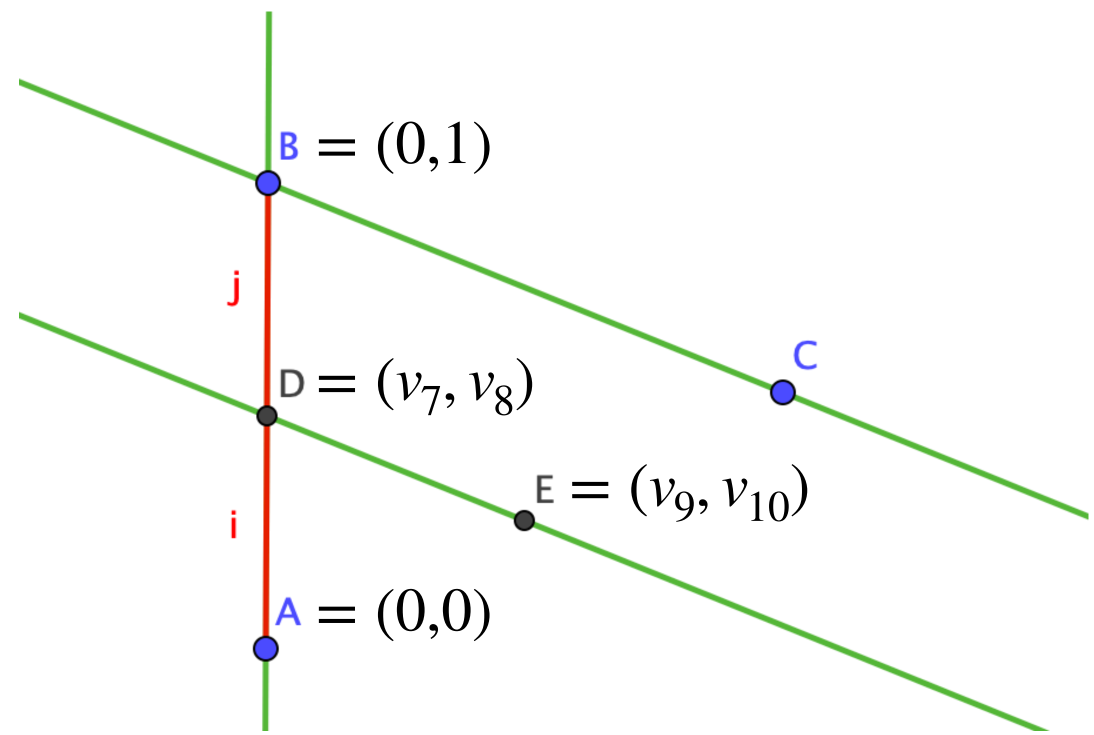


New Result

E is the midpoint of AC
D is aligned with A and B
BC is parallel to DE

If E is not aligned with A and B $\Longrightarrow$ D is the midpoint of AB $(i = j)$

$v_9 \neq 0$

''converse'' of the Midpoint Theorem

Difficulty degree = 3

Derivation 2

New Thesis

New Hypotheses

$$s_1 \cdot (v_{10} + 1/2) = T - s_2 \cdot v_9 - s_3 \cdot (v_{10} - v_8) - s_4 \cdot (v_7 - v_9)$$

Derivation 2

New Thesis

New Hypotheses

$$s_1 \cdot (v_{10} + 1/2) = T - s_2 \cdot v_9 - s_3 \cdot (v_{10} - v_8) - s_4 \cdot (v_7 - v_9)$$

$D \in$ line AB

$DE \,//\, BC$

$D \in$ perp. bisec. **p** of AB

E is the midpoint of AC

Derivation 2

New Thesis

New Hypotheses

$$s_1 \cdot (v_{10} + 1/2) = T - s_2 \cdot v_9 - s_3 \cdot (v_{10} - v_8) - s_4 \cdot (v_7 - v_9)$$

$D \in$ line AB

$DE \,//\, BC$

$D \in$ perp. bisec. **p** of AB

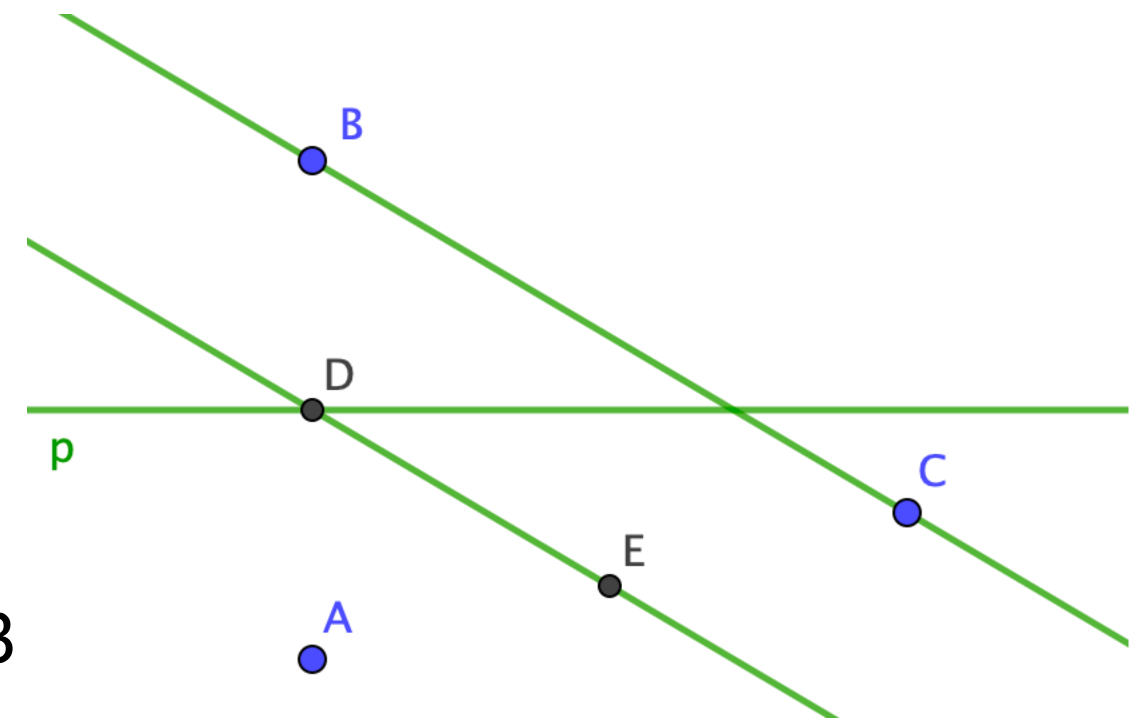
E is the midpoint of AC

New Result

E midpoint of AC
D is in the perpendicular bisector ***p*** of AB
BC is parallel to DE

If E is not in ***p*** $\Longrightarrow$ D is aligned with A and B

$v_{10} + 1/2 \neq 0$



Difficulty degree = 4

Final comments and future directions

This is an initial work.

- Automatic derivation not yet implemented but promising.
- Definition of difficulty/complexity of a statement is not yet settled:
  - Monomial order dependence?
  - Is there a better algorithm not using Gröbner basis?
  - There is one that guarantees computing the lowest degree of a_i?

$$T^k = a_1' s_1 + \ldots + a_n' s_n, \text{ for some } k$$

- Comparing difficulty of humans / AI bots / GeoGebra Discovery
- Extending the method to real equations and inequalities

Thanks for the attention!!!

References

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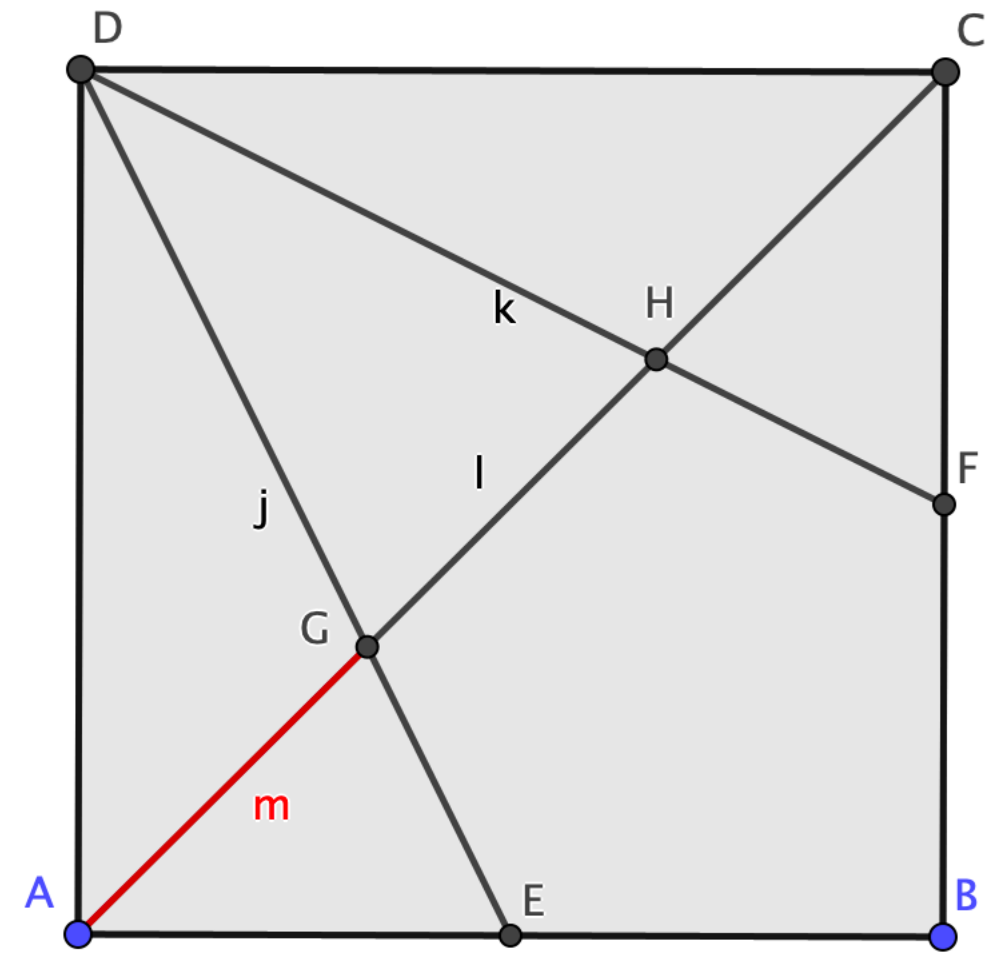
Problem from “Kuwait” ICMI Study (1986):

Given the following construction, the diagonal AC is divided in 3 segments.

Are those segments equal?

Equivalently, is $l = 3m$?



Play with the commands:

Relation(3m == l)
Discovery(F)

Difficulty degree = 5